A CLASS OF EXACT SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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Let us consider a steady axisymmetric flow of a viscous incompressible fluid. It is described by the following equations in a cylindrical coordinate system:

$$v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial r v_r}{\partial r} + \frac{\partial^2 v_r}{\partial z^2}\right),$$
$$\frac{v_r}{r}\frac{\partial r v_\varphi}{\partial r} + \frac{v_z}{r}\frac{\partial r v_\varphi}{\partial z} = \nu\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial r v_\varphi}{\partial r} + \frac{\partial^2 v_\varphi}{\partial z^2}\right),$$
$$v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2}\right),$$
$$\frac{\partial r v_r}{\partial r} + \frac{\partial r v_z}{\partial z} = 0\ . \qquad (1)$$

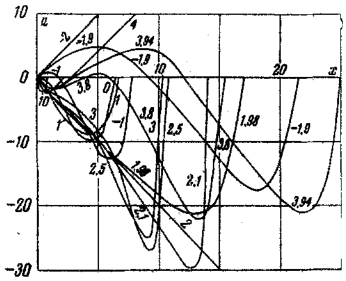

Fig. 1

We shall seek the solution in the form $v_\varphi = v_\varphi(r) \not\equiv 0$, then $v_r = v_r(r)$, $v_z = zw(r)$. Substituting these expressions in (1), we obtain

$$v_r \frac{dv_r}{dr} - \frac{v_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu \frac{d}{dr}\frac{1}{r}\frac{drv_r}{dr}, \qquad (2)$$
$$\frac{v_r}{r}\frac{drv_\varphi}{dr} = \nu \frac{d}{dr}\frac{1}{r}\frac{drv_\varphi}{dr}, \qquad (3)$$
$$zv_r \frac{dw}{dr} + zw^2 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + z\frac{\nu}{r}\frac{d}{dr} r \frac{dw}{dr}, \qquad (4)$$
$$\frac{drv_r}{dr} + rw = 0\ . \qquad (5)$$

Equations (3) and (4) imply that $\rho^{-1}\partial p/\partial r = F(r)$, and $\rho^{-1}\partial p/\partial z = zf(r)$. We find from this that it is necessary to set

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -4\delta a^2 z, \qquad \delta = \pm 1, \qquad a = \text{const}. \qquad (6)$$

Substituting (6) in (4), we obtain

$$v_r \frac{dw}{dr} + w^2 = 4\delta a^2 + \frac{\nu}{r}\frac{d}{dr} r \frac{dw}{dr}. \qquad (7)$$

Equations (5) and (7) form a closed system which can be solved independently of equations (2) and (3). The latter serve for determining the functions $p(r, z)$ and $v(r)$ after finding $v_r(r)$. Thus, v_r and w do not depend on whether the flow is twisted or not. We set

$$rv_r = -\nu u, \qquad x = ar^2 / 2\nu.$$

Then it is easy to reduce equation (5) to the form

$$w = adu / dx. \qquad (8)$$

After simple transformations of equation (7), taking (8) into consideration, we obtain a single third-order equation not containing the parameter

$$2(xu'')' = u'^2 - uu'' - 4\delta. \qquad (9)$$

Here the primes indicate differentiation with respect to x. After introducing the function $\Phi = rv_\varphi$, equation (3) is transformed to the form

$$2x\Phi'' + u\Phi' = 0. \qquad (10)$$

We shall seek a one-parameter family of solutions of equation (9) depending on the parameter m and satisfying the conditions

$$u(0) = 0, \qquad u'(0) = -m, \quad u''(0) \text{ is bounded}$$

It is not difficult to see that in order to satisfy the last condition, we must set

$$u''(0) = {}^1/_2\,(m^2 - 4\delta). \qquad (11)$$

We shall consider two cases.

First case $\delta = 1$. Exact solutions are found for five values of m. (The solutions for the cases m = 2 and m = 4 are known [1, 2]).

$$u \mp = 2x \qquad (m = \pm 2)$$
$$u = \pm 2x - 6(1 - e^{\mp x}) \qquad (m = \pm 4)$$
$$u(\xi) = \xi + \sum_{n=1}^{\infty} \frac{\xi^n}{2^k n!}, \qquad k = \frac{6n - 1 + (-1)^n}{4} \qquad (m = \infty).$$

The substitution $\xi = mx$ was introduced for the case $m \to \infty$. Equation (9) takes the form

$$2(\xi u'')' = u'^2 - uu'' - \frac{4}{m^2},$$
$$u(0) = 0, \ u'(0) = -1, \ u''(0) = \frac{1}{2}\left(1 - \frac{4}{m^2}\right).$$

The solutions for the remaining values of m were obtained numerically with the aid of a M-20 electronic computer. A family of integral curves is shown in Fig. 1. The curves were constructed to values of x^* corresponding to the last root of the function u(x). When $x > x^*$, a sharp increase in the function u(x) was observed in all cases up to values exceeding 10^{19}. The calculations carried out for the function $s(x) = u^{-1}$ showed that at some $x^{**} > x^*$, the function s(x) changes sign. This means that the function u(x) is without bounds in the neighborhood of the point x^{**}.

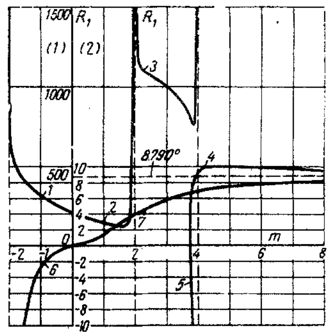

Fig. 2

The integral curves corresponding to values of $m = \pm 2$ and $m = 4$ are in a certain sense special, for on approaching these values, the corresponding curves approach the limit values, but not uniformly. For example, for any value $\varepsilon > 0$, when $m = 2 \pm e$, u(x) will be larger than zero, beginning with some x, even though $u(x) = -2x$ when m = 2. When $m \le -2$, the integral curves will have no roots. It is important to note that when $3.782 < m < 4$, each curve has three roots, not counting x = 0; when $-2 < m < 0$, each curve has two roots; and in the other cases, one root.

Second case $\delta = -1$. The integral curves form a monotonic family $u_k(x) > u_l(x)$ if $k < l$, where k and l are the respective values of the parameter m. When $m < 0$, all curves have one root apiece; when $m < 0$, there are no roots.

Let us turn to equation (10). Its solution which satisfies the condition $\Phi(0) = 0$ is of the form

$$\Phi(x) = \Phi'(0)\int_0^x \exp\left(-\int_0^x \frac{udx}{2x}\right)dx\ , \qquad (12)$$

$$\Phi = \pm \Phi'(0)(e^{\pm x} - 1) \quad \text{at} \quad \delta = 1, \quad m = \pm 2\,. \tag{12}$$
(cont'd)

For other values of m, the function $I(x) = \Phi/\Phi'(0)$ was computed simultaneously with u(x).

We shall consider the following two problems for a hydrodynamic interpretation of the class of solutions of the Navier-Stokes equations we have found here.

Problem A. This work was done as a result of attempts to develop a model of the flow in the axial zone of a vortex chamber. The latter was a cylinder with a coaxial opening in one of the bottoms. Slits were cut along several generatrices of the cylinder through which fluid was introduced tangentially into the chamber. It was established [3] that an almost cylindrical region with radius r_0 was formed in the vicinity of the axis of the chamber which was not entered by the initially introduced fluid. When liquid droplets were discharged into the air, the surface $r = r_0$ was a surface of discontinuity with an "air vortex" inside it. When air was discharged into this region, there was no discontinuity, but the surface $r = r_0$ remained impermeable to the main flow. Although the flow in the region $r > r_0$ could be described approximately with the aid of the scheme for an ideal fluid, no theoretical model has been developed as yet for the region $r < r_0$, where there are complicated secondary flows.

As established in reference [3], the axial speed v_z varies almost linearly along the surface $r = r_0$ on which $v_r = 0$; $v_z = kz$, and the tangential speed remains approximately constant $v_\varphi = v_0$ along the surface $r = r_0$ on which $v_r = 0$.

These special characteristics of flow in a vortex chamber provide a basis for attempting to construct the following model of flow in the region $r < r_0$. We consider a semi-infinite pipe with radius r_0 and seek motion of a viscous fluid satisfying the conditions

$$\begin{gathered} v_r = 0, \quad v_z = kz, \quad v_\varphi = v_0 \\ \text{at} \quad r = r_0; \quad v_z = 0 \quad \text{when } z = 0\,, \\ v_r = v_\varphi = 0, \quad v_z < \text{const} \quad \text{when } r = 0\,. \end{gathered} \tag{13}$$

No conditions at infinity or no-slip conditions at the end $z = 0$ are specified. The latter can be justified by the fact that we are considering either flow outside the end boundary layer or a symmetric vortex chamber with outlets in both bottoms so that the center plane in such a chamber will be impermeable, but "absolutely smooth," and the no-slip conditions are replaced by the symmetry conditions

$$\begin{gathered} v_r(r, z) = v_r(r, -z), \qquad v_\varphi(r, z) = v_\varphi, \\ (r, -z), \qquad v_z(r, z) = -v_z(r, -z)\;. \end{gathered} \tag{14}$$

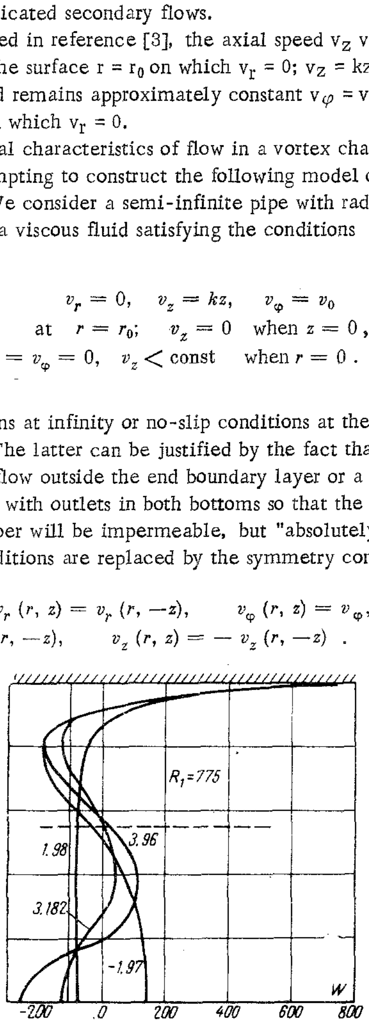

Fig. 3

We shall attempt to subordinate the solutions of equation (9) to conditions (13) and (14) by suitable choice of the parameter m. It is obvious that all conditions, except $w(r_0) = k$ and $v(r_0) = r_0$, are automatically satisfied. Let $x_0 = 1/2\, a r_0^2/\nu$ be some root of the function u(x) and $N = u'(x_0)$. The condition $w(r_0)$ together with (8) leads to the relationship $aN = k$; consequently,

$$a = \frac{2\nu x_0}{r_0^2}\,, \qquad R_1 = \frac{k r_0^2}{\nu} = 2x_0 N\,. \tag{15}$$

Here R_1 is the Reynolds number characteristic of problem A. The right-hand side of (15) depends only on the number of the root N and m. Thus, we can construct the relationship $R_1(m)$ which is shown in Fig. 2. The function $R_1(m)$ consists of seven branches which correspond to the following values of the parameters:

Branch	1	2	3	4	5	6	7
δ =	1	1	1	1	1	1	−1
N =	2	1	3	1	2	1	1
Interval	(−2,0)	(0, 2)	(2, 4)	(3.782, ∞)	(3.782,4)	(−2,0)	(0, ∞)

Branches 1, 2, and 3 are constructed on scale (1) in the figure and the remaining branches on scale (2). Negative values of R_1 correspond to values of $k < 0$. The most characteristic property of the constructed solutions can be seen in Fig. 2—their lack of identity. With values $-\infty < R_1 < 10.2$, there are two solutions: in the interval $10.2 < R_1 < 142$, problem A has no solutions of the given class. When $142 < R_1 < 731$, there are again two solutions, and, finally, when $R_1 > 731$, the number of solutions reaches four. Fig. 3 shows curves of the dimensionless function $W = wr_0^2/\nu$ versus r/r_0 corresponding to the value $R_1 = 775$ where the problem has four solutions. The calculations were carried out on the basis of the equalities

$$r/r_0 = \sqrt{x/x_0}, \qquad wr_0^2/\nu = 2x_0 u\,.$$

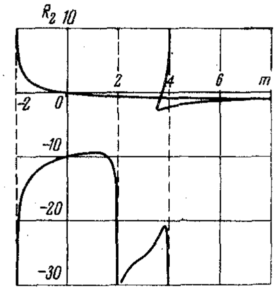

Fig. 4

The latter is a consequence of (8).

The existence of two solutions when $R_1 = 0$ is wholly logical from the physical standpoint if we consider the method for computing R_1. The fact is that movement with $N = 0$ can be given a dual interpretation. In the first place, it can be regarded as independent movement in a pipe with a fixed wall. This case corresponds to the value $m = 0$ and is characterized by the fluid at rest. In the second place, the given flow can be regarded as representing a component of the flow with $m = 3.782$ and $R_1 = 775$ and developing within the region $r < r^*$ (the dashed line of Fig. 3). If the surface $r = r^*$ on which $v_r = v_z = 0$ is replaced by a solid wall, then within the region $r \le r^*$, the fluid should be physically at rest; if this is not done, there should be motion since $dw/dr \ne 0$ when $r = r^*$, and tangential forces of the external part of the flow will act on the fluid in the region $r \le r^*$.

Motion of the fluid in the pipe was caused by the drag effect of the wall, thus, only those solutions of equation (9) that satisfy the condition $dw/dr > 0$ when $r = r_0$ can be solutions to problem A. In accordance with this requirement, a part of branch 4 in Fig. 2 for values $3.782 < m < 3.92$ should be discarded. Then problem A will have a unique solution corresponding to branch 7 on the interval $0 < R_1 < 8.5$ We note that these solutions correspond to motion with a positive pressure gradient $\partial p/\partial z > 0$. It is interesting to note that all four types of motion shown in Fig. 3 have been observed experimentally, under different conditions, of course.

Problem B. Let us consider a semi-infinite pipe with radius r_0 with a fized porous wall. Let a fluid be injected through the lateral surface of the pipe uniformly along its entire length at an injection velocity of $v_{r_0} = -\nu u_0/r_0$. If we again do not specify a condition of attachment at the end, but require that $v_z(r, 0) = 0$, the problem is reduced to seeking those solutions of equation (9) which satisfy the conditions $u(0) = 0$; $u'(0)$ is bounded; $u(x_0) = u_0$, where x_0 is the root of the function $u'(x)$. The dimensionless parameter $u_0 = -r_0 v_{r_0}/\nu$ plays the role of the Reynolds number characteristic of problem B. Positive values of u_0 correspond to injection of fluid into the pipe and negative values, suction of fluid from the pipe. The relationship $R_2 = u_0(m)$ is shown in

Fig. 4. If we require that $v_z \geq 0$, then when $R_2 \geq 0$, problem B will have a unique solution corresponding to negative values of m.

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